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Determination of the sonic line in hypersonic flow past a blunt body

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The Newtonian theory of inviscid hypersonic flow is extended to obtain a solution uniformly valid in the subsonic region, and that is used to determine the position and shape of the sonic line. The main modification to the theory has to be made near the body surface and an expansion, essentially in terms of the stream function, is employed.

For simplicity the solution is limited to the cases of axially- and plane-symmetric flows. As an illustration of the theory the flows past a sphere and a circular cylinder are treated in some detail. Comparison with the numerical results of Garabedian and Lieberstein gives favourable agreement.

1. Introduction

In hypersonic flow past a blunt body the incident stream is arrested by the strong shock which appears in front of the body. When the body is blunt and symmetrical about its axis the shock is also symmetrical and crosses the axis at right angles a short distance ahead of the body. The flow between the shock and the body is subsonic in the neighbourhood of the axis where the fluid has crossed an almost normal shock, but it speeds up as it moves away from the axis and crosses the sonic line into the supersonic region that surrounds the part of the body away from the nose. Approximate solutions for hypersonic flow past a blunt body can be determined by using the Newtonian approximation, which is valid when the parameter $\epsilon = (\gamma-1)/(\gamma+1)$ is small. The expansion procedures used in these solutions lead to the results that the velocity component parallel to the body and the speed of sound are each constant along a streamline (see Hayes & Probstein 1959). Though the latter result is uniformly valid, the former breaks down in the neighbourhood of the body surface since the streamlines there come from the stagnation region. However, near the shock the two results can be used to determine the shape of the sonic line, which initially follows the streamline through the sonic point on the shock.

In this paper a solution valid in the subsonic region will be presented; the limitation to the subsonic region allows certain simplifications to be made. The flow variables are found in terms of the pressure on the body, which is correctly given to a first approximation by Freeman's solution (1956). Alternatively, this pressure distribution could be obtained from experiment.

Numerical calculations of the position of the sonic line have previously been

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obtained by Garabedian and Lieberstein by considering the indirect problem, in which a certain shock shape is assumed and the corresponding body shape is calculated. An example of their calculations is given by Hayes & Probstein for a free-stream Mach number of 20 and a gas for which the ratio of specific heats behind the shock is 1·17. Reasonable agreement with their solution is obtained.

2. Conditions at the shock

Let the pressure, density and enthalpy be denoted by p, ρ and i respectively, and let the suffix 0 refer to the uniform conditions ahead of the shock and the suffix s to conditions immediately behind the shock. Let U_0 be the uniform velocity of the incident stream and let u_{ns} and u_{ts} be the components of velocity normal and tangential to the shock and immediately behind it. Then, at a point where the shock is inclined at an angle Θ to the incident stream, the Rankine–Hugoniot equations give

$$\rho_0 U_0 \sin\Theta = \rho_s u_{ns}, \tag{2.1}$$

$$p_0 + \rho_0 U_0^2 \sin^2\Theta = p_s + \rho_s u_{ns}^2, \tag{2.2}$$

and

$$i_0 + \tfrac{1}{2}U_0^2 \sin^2\Theta = i_s + \tfrac{1}{2}u_{ns}^2. \tag{2.3}$$

In addition the tangential component of velocity is continuous so that

$$U_0 \cos\Theta = u_{ts}. \tag{2.4}$$

In hypersonic flow the free-stream Mach number is large so that the usual strong shock approximations can be made; the equations (2.2) and (2.3) then simplify and become

$$\rho_0 U_0^2 \sin^2\Theta = \rho_s u_{ns}^2 + p_s, \tag{2.5}$$

and

$$\tfrac{1}{2}U_0^2 \sin^2\Theta = i_s + \tfrac{1}{2}u_{ns}^2. \tag{2.6}$$

For a perfect gas the density ratio across the shock is

$$\rho_0/\rho_s = (\gamma-1)/(\gamma+1) = \epsilon. \tag{2.7}$$

In terms of ϵ the flow quantities immediately behind the shock are

$$u_{ns} = \epsilon U_0 \sin\Theta, \tag{2.8}$$

$$p_s = (1-\epsilon)\rho_0 U_0^2 \sin^2\Theta, \tag{2.9}$$

and

$$i_s = \tfrac{1}{2}(1-\epsilon^2)\, U_0^2 \sin^2\Theta. \tag{2.10}$$

3. The sonic point at the shock

From equation (2.10) the speed of sound is given by

$$a_s^2 = \epsilon(1+\epsilon)\, U_0^2 \sin^2\Theta. \tag{3.1}$$

The shock angle Θ^* at the sonic point can now be determined with the help of expressions (2.4), (2.8) and (3.1), and it is given by

$$\tan^2\Theta^* = 1/\epsilon. \tag{3.2}$$

If $\tfrac{1}{2}\pi - \Theta^* = \theta^*$, it follows that

$$\theta^* \simeq \epsilon^{\frac{1}{2}}. \tag{3.3}$$

4. Method of solution

Previous work in this field shows that near the shock both the velocity component parallel to the body and the speed of sound are constant along streamlines so that a first approximation to the sonic line is given by the streamline through the sonic point on the shock. It is known that the actual sonic line deviates towards the body from the streamline (see Hayes & Probstein 1959) so that the region of interest is a region covered by the streamlines crossing the shock between the stagnation point and the sonic point on it.

A co-ordinate system is chosen in which x is measured along the body from the stagnation point and y perpendicular to it; u and v denote the velocity components in the x and y directions, respectively. The continuity equation is

$$\frac{\partial}{\partial x}(\rho u k^n) + \frac{\partial}{\partial y}(\rho v k^n[1+y/R_b]) = 0, \tag{4.1}$$

where $(1+y/R_b)\,dx$ is an element of length in the x-direction, $R_b(x)$ is the radius of curvature of the body, and k is the distance of a point from the line of symmetry, which is taken to mean the plane of symmetry in the two-dimensional case and the axis of symmetry in the axially symmetric case. In the two-dimensional case $n = 0$ and for the case of axial symmetry $n = 1$. After introducing a stream function ψ defined by

$$\partial\psi/\partial y = \rho u k^n \quad \text{and} \quad \partial\psi/\partial x = -\rho v k^n(1+y/R_b), \tag{4.2}$$

the variables are changed from x and y to x and ψ.

Conditions at the shock give

$$\psi/\rho_0 U_0 = [k_s(\xi)]^{n+1}/(n+1), \tag{4.3}$$

where k_s is the distance of a point on the shock from the line of symmetry, and ξ is the x-co-ordinate of the point where the streamline crosses the shock.

As a first approximation the shock is assumed to be parallel to the body in the stagnation region, though some relaxation of this condition could be made that involves the solution of a certain integral equation. Freeman showed that the stand-off distance is of the order of $(\epsilon \log \epsilon)d$ in the two-dimensional case and ϵd in the axially symmetric case so that the radius of curvature of the shock is of order d where d is the radius of curvature of the body at the nose and will be used as a representative length. Now

$$k_s(\xi) = \int_0^{\theta(\xi)} R_s \cos\theta \, d\theta, \tag{4.4}$$

which, for the region under consideration where $\theta = O(\epsilon^{\frac{1}{2}})$ and assuming

$$d^{-1}\,dR_s/d\theta = O(1),$$

yields the approximate expression

$$k_s(\xi) = \xi + O(\epsilon d). \tag{4.5}$$

Equation (4.3) may therefore be written

$$\psi/\rho_0 U_0 = \xi^{n+1}/(n+1). \tag{4.6}$$

If x and ξ are now used as independent variables, the momentum equation is

$$\frac{k^n}{\xi^n}\frac{\partial p}{\partial \xi}+\frac{R_b}{R_{b+y}}\frac{\partial v}{\partial x}-\frac{u}{R_{b+y}} = 0. \tag{4.7}$$

Bernoulli's equation is

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\tfrac{1}{2}(u^2+v^2) = \tfrac{1}{2}U_0^2, \tag{4.8}$$

and also

$$p/\rho^\gamma = p_s(\xi)/[\rho_s(\xi)]^\gamma. \tag{4.9}$$

Substituting equations (2.7) and (2.9) for ρ_s and p_s respectively in equation (4.9), a series expansion in ϵ is obtained for p/ρ and involves $\log(p/\rho_0 U_0^2)$. This, when substituted in equation (4.8) and when terms of $O(\epsilon^2)$ are neglected, yields the equation

$$2\epsilon\log(p/\rho_0 U_0^2)-\theta^2(\xi)+(u/U_0)^2 = 0, \tag{4.10}$$

since $v = O(\epsilon U_0)$ from conditions at the shock. The form of equation (4.10) with the condition that $\theta = O(\epsilon^{\frac{1}{2}})$ suggests that an expansion in terms of $\epsilon^{\frac{1}{2}}$ in the form

$$p/\rho_0 U_0^2 = p_0+\epsilon^{\frac{1}{2}}p_1+\epsilon p_2+\ldots, \tag{4.11}$$

$$u/U_0 = \epsilon^{\frac{1}{2}}u_1+\epsilon u_2+\ldots, \tag{4.12}$$

should be tried.

In the region under consideration the unit of length in the ξ direction is $\epsilon^{\frac{1}{2}}d$, so that ξ is replaced by $\xi_1 = \xi/\epsilon^{\frac{1}{2}}d$.

Substituting for ξ and the above expansions in equation (4.7) and then equating coefficients it is found that

$$p_0 = p_0(x) \quad \text{and} \quad p_1 = p_1(x), \tag{4.13}$$

since the second and third terms in the equation are $O(\epsilon)$ and $O(\epsilon^{\frac{1}{2}})$ respectively whereas the first term is

$$\frac{k^n}{(\epsilon^{\frac{1}{2}}d)^{n+1}}\sum_{r=0}\frac{\partial p_r}{\partial \xi_1}\epsilon^{\frac{1}{2}r}.$$

In obtaining the above result (4.13) it must be remembered that near the shock in axially symmetric flow $k^n = O(\epsilon^{\frac{1}{2}}d)$. Hence, in the subsonic region the pressure is a function of x only to $O(\epsilon^{\frac{1}{2}})$. If the value of the pressure on the body is $\rho_0 U_0^2 p_b(x)$ say, then

$$p_0+\epsilon^{\frac{1}{2}}p_1 = p_b(x). \tag{4.14}$$

Using Bernoulli's equation and putting $\theta = \phi$, where ϕ is the complement of the angle of incidence of the body (using Freeman's analysis this can be shown to be consistent with the approximations used here), the first approximation for the velocity component u is obtained in the form

$$u = U_0\epsilon^{\frac{1}{2}}u_1 = U_0[\phi^2(\xi)-2\epsilon\log p_b(x)]^{\frac{1}{2}}. \tag{4.15}$$

Hence, using expression (4.15) and the fact that the speed of sound is constant along a streamline and takes the value $U_0\epsilon^{\frac{1}{2}}$ as given by equation (3.2), the first approximation to the sonic line is

$$[\phi^2(\xi)-2\epsilon\log p_0(x)]^{\frac{1}{2}} = \epsilon^{\frac{1}{2}}. \tag{4.16}$$

For the sonic line near the shock $x = O/(\epsilon^{\frac{1}{2}}d)$ and $p_b(x) = 1 + O(\epsilon)$. Equation (4.16) then shows that the first term in brackets makes the greater contribution to the velocity and gives the sonic line to be the streamline through the sonic point on the shock. Near the body, where $x = O(d)$, $p_b(x)$ is considerably less than unity, $\phi(\xi)$ is tending to zero and so the second term becomes important.

The error involved in equation (4.16) is $O(\epsilon)$. Since in deriving equation (4.10) terms of $O(\epsilon^2)$ were neglected, the equation can be used to discuss the magnitude of u_2. It is seen that u_2 is entirely dependent on the slope σ of the shock relative to the body since $\theta = \phi + \sigma$ and if $\sigma = O(\epsilon^{\frac{3}{2}})$ then $u_2 \equiv 0$. In this case the error in the equation for the sonic line is $O(\epsilon^{\frac{3}{2}})$.

The equation of the streamlines may now be found using equation (4.2) and noting from equation (4.9) and the conditions at the shock that

$$p/\rho U_0^2 = \epsilon + O(\epsilon^2),$$

The streamlines are given by

$$y = \int_0^{\psi} \frac{d\psi}{\rho u k^n},$$

using equation (4.2), and so

$$y = \epsilon \int_0^{\xi} \frac{t^n}{p_b(x)\, k_b^n(x)\, [\phi^2(t) - 2\epsilon \log p_b(x)]^{\frac{1}{2}}}\, dt, \tag{4.17}$$

since $k^n = k_b^n + O(\epsilon d)$. At the shock $\xi = x$ and hence the equation of the shock is simply obtained by substituting x for ξ in the above equation.

The equation for the sonic line is obtained in terms of x and y by substituting for ξ from equation (4.17) in equation (4.16).

5. The second approximation when $\sigma = O(\epsilon^{\frac{3}{2}})$.

From the conditions at the shock the velocity component v can be seen to be $O(\epsilon U_0)$ so that substituting the expansions for the pressure and the velocity component u in equation (4.7) gives the next approximation of the momentum equation

$$\frac{k_b(x)}{\xi^n} \frac{\partial p_2}{\partial \xi} - \frac{u_1}{R_b} = 0. \tag{5.1}$$

By integrating equation (5.1) and using the known expression for u_1 the equation

$$p_2 = p_{2b}(x) + \frac{1}{k_b^n(x)\, R_b(x)} \int_0^{\xi} [\phi^2(t) - 2\epsilon \log p_b(x)]^{\frac{1}{2}}\, t^n\, dt, \tag{5.2}$$

is obtained. The substitution for ρ from equation (4.9) into equation (4.8) gives on expanding and neglecting terms of $O(\epsilon^{\frac{5}{2}})$

$$\begin{aligned} -\phi^2(\xi) + 2\phi(\xi)\,\sigma(\xi) + 2\epsilon \log p_b(x) + \epsilon^2 + 2\epsilon\phi^2(\xi) - 2\epsilon^2 \log p_b(x) \\ + 2\epsilon^2[\log p_b(x)]^2 + 2\epsilon^2 p_2(x,\xi)/p_b(x) - 2\epsilon\phi^2(\xi) \log p_b(x) + q^2 = 0, \end{aligned} \tag{5.3}$$

from which the total velocity q can be obtained.

Freeman's first approximation to the velocity component u parallel to the body, namely $u = u_0 \cos \Theta(\xi)$, is certainly valid in the region between the shock and the streamline through the sonic point on the shock since there the values of

$\cos\Theta(\xi)$ and u/U_0 are both $O(\epsilon^{\frac{1}{2}})$ but in the subsonic region Freeman's assumption that u is constant along each streamline, is not valid since $\cos\Theta(\xi) \to 0$ as $\xi \to 0$ whereas u/U_0 may be $O(\epsilon^{\frac{1}{2}})$.

Hence in order to evaluate $p_{2b}(x)$ the expression for the pressure distribution near the body, i.e. $p_b + \epsilon p_2$, must be matched with Freeman's solution, taken to a second approximation, across the streamline through the sonic point on the shock, i.e. where $\xi = \epsilon^{\frac{1}{2}}$.

The speed of sound to $O(\epsilon^2)$ is given by

$$a^2 = \epsilon + \epsilon^2[2\log p_b(x) - 1 - \phi^2(\xi)], \tag{5.4}$$

and thus would enable the equation for the sonic line on which $q = a$ to be obtained. A more detailed analysis of the flow might proceed as follows. From the known value of u_1 the first approximation to the velocity component v can be found by using equation (4.2) and hence the value of u_3 can be determined from equation (5.3). This permits the third approximation p_3 for the pressure to be obtained provided there exists a sufficiently accurate solution near the shock with which to match the expression thus obtained.

6. Solutions for the sphere and circular cylinder

Of particular interest are the cases of the sphere in axially symmetric flow and the circular cylinder in plane symmetric flow. For both of these cases

$$\phi(x) = x/d, \tag{6.1}$$

and

$$k_b = R_b \sin x', \tag{6.2}$$

where $x' = x/d$.

Expressions for the pressure distribution on the body in two cases are now required, and these are simply those obtained by Freeman, since, although his analysis is only valid near the shock, the pressure is constant to $O(\epsilon^{\frac{1}{2}})$ across the layer near the body, as has already been seen. The pressure distributions on the body are therefore given by

$$\left.\begin{aligned} &\text{(i)}\ \ p_b(x) = 1 - \tfrac{4}{3}\sin^2 x' \quad \text{for the sphere,} \\ \text{and}\quad &\text{(ii)}\ \ p_b(x) = 1 - \tfrac{3}{2}\sin^2 x' \quad \text{for the cylinder.} \end{aligned}\right\} \tag{6.3}$$

The equations of the streamlines are now obtained by the use of the above expressions, giving

$$\left.\begin{aligned} &\text{(i)}\ \ \frac{y}{d} = y' = \frac{\epsilon}{\sin x'(1 - \frac{4}{3}\sin^2 x')}\{[\xi'^2 - 2\epsilon\log(1 - \tfrac{4}{3}\sin^2 x')]^{\frac{1}{2}} \\ &\qquad\qquad - [-2\epsilon\log(1 - \tfrac{4}{3}\sin^2 x')]^{\frac{1}{2}}\} \\ \text{and}\quad &\text{(ii)}\ \ y' = \frac{\epsilon}{1 - \frac{3}{2}\sin^2 x'}\sinh^{-1}\left\{\frac{\xi'}{[-2\epsilon\log(1 - \frac{3}{2}\sin^2 x')]^{\frac{1}{2}}}\right\}, \end{aligned}\right\} \tag{6.4}$$

where $\xi' = \xi/d$.

The equations of the sonic lines in terms of x and y are then determined, namely

$$\left.\begin{aligned} &\text{(i)}\ \ \sin x'(1 - \tfrac{4}{3}\sin^2 x')y'/\epsilon + [-2\epsilon\log(1 - \tfrac{4}{3}\sin^2 x')]^{\frac{1}{2}} = \epsilon^{\frac{1}{2}}, \\ \text{and}\quad &\text{(ii)}\ \ \cosh\{(1 - \tfrac{3}{2}\sin^2 x')\,y'/\epsilon\} = 1/[-2\log(1 - \tfrac{3}{2}\sin^2 x')]^{\frac{1}{2}}. \end{aligned}\right\} \tag{6.5}$$

Examination of the second equation in (6.5) shows that in the two-dimensional case the sonic line is orthogonal to the body surface. Also in both of these cases the difference in slope between the shock and body surface is $O(\epsilon^{\frac{1}{2}})$, so that $u_2 \equiv 0$ and hence equations (6.5) are accurate to $O(\epsilon)$. Figure 1 shows the shapes of the

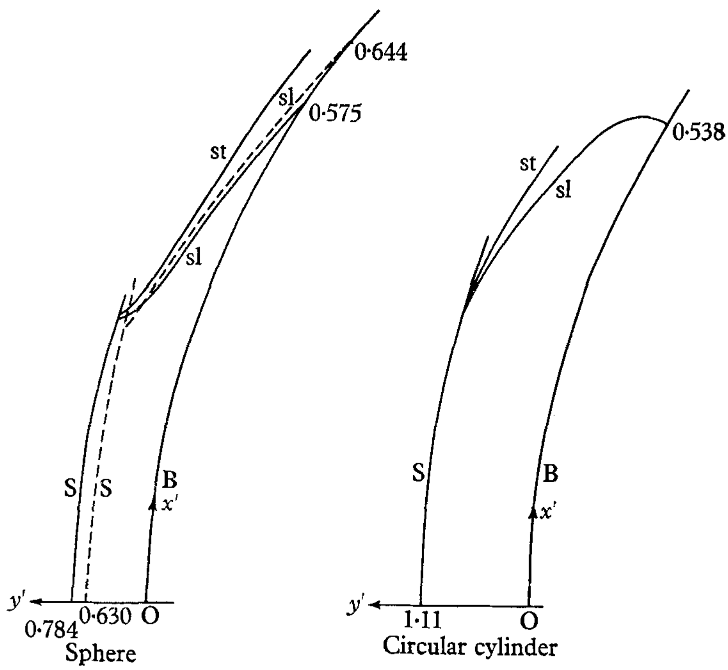

FIGURE 1. Sonic lines for a sphere and a circular cylinder with $\gamma_s = 1{\cdot}17$. S = shock; B = body; st = streamline; sl = sonic line. (The broken lines refer to the results of Garabedian and Lieberstein).

two sonic lines and in the case of the sphere can be compared with the unpublished results of Garabedian and Lieberstein (Hayes & Probstein 1959).

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